

BOUNDING THE VOLUMES OF SINGULAR FANO THREEFOLDS

CHING-JUI LAI

ABSTRACT. Let (X, Δ) be an n -dimensional ϵ -klt log \mathbb{Q} -Fano pair. We give an upper bound for the volume $\text{Vol}(-(K_X + \Delta)) = -(K_X + \Delta)^n$ when $n = 2$ or $n = 3$ and X is \mathbb{Q} -factorial of $\rho(X) = 1$. This bound is essentially sharp for $n = 2$. Existence of an upper bound for anticanonical volumes is related the Borisov-Alexeev-Borisov Conjecture which asserts boundedness of the set of ϵ -klt log \mathbb{Q} -Fano varieties of a given dimension n .

Throughout this article, we work over field of complex numbers \mathbb{C} . We recall the definition of singularities of pairs and log \mathbb{Q} -Fano pairs.

Definition 0.1. A pair (X, Δ) consists of a normal projective variety X and a boundary Δ , i.e., a \mathbb{Q} -divisor Δ with coefficients in $[0, 1]$, such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $\pi : Y \rightarrow X$ be a log resolution of (X, Δ) , the discrepancy $a(E, X, \Delta)$ of a divisor E on Y with respect to the pair (X, Δ) is defined by $a(E, X, \Delta) = \text{mult}_E(K_Y - \pi^*(K_X + \Delta))$. We say that (X, Δ) has only terminal (resp. canonical) singularities if $a(E, X, \Delta) > 0$ (resp. ≥ 0) for any π -exceptional divisor E on Y . We say that (X, Δ) is klt (resp. ϵ -klt for some $0 < \epsilon < 1$) if $a(E, X, \Delta) > -1$ (resp. $> -1 + \epsilon$) for any divisor E on Y . Note that smaller ϵ corresponds to worse singularities.

A pair (X, Δ) is (weak) log \mathbb{Q} -Fano if the \mathbb{Q} -Cartier divisor $-(K_X + \Delta)$ is ample (resp. nef and big).

For a klt pair (X, Δ) with $\kappa(K_X + \Delta) = -\infty$, according to the log minimal model program, there exists a birational map $\phi : X \dashrightarrow Y$ and a morphism $Y \rightarrow Z$ such that for $\Delta' = \phi_*\Delta$, the pair (Y_z, Δ'_z) is log \mathbb{Q} -Fano with $\rho(Y_z) = 1$ for general $z \in Z$. In particular, log \mathbb{Q} -Fano pairs are the building blocks for pairs with negative Kodaira dimension. It is also expected that the set of mildly singular \mathbb{Q} -Fano varieties is bounded.

Definition 0.2. We say that a collection of varieties $\{X_\lambda\}_{\lambda \in \Lambda}$ is bounded if there exists $h : \mathcal{X} \rightarrow S$ a morphism of finite type of Noetherian schemes such that for each X_λ , $X_\lambda \cong \mathcal{X}_s$ for some $s \in S$.

For example, the set of all the n -dimensional smooth Fano manifolds is bounded by [17]. Boundedness is also known for terminal \mathbb{Q} -Fano \mathbb{Q} -factorial threefolds of Picard number one by [12] and for canonical \mathbb{Q} -Fano threefolds by [18]. However, if one considers the set of all klt \mathbb{Q} -Fano varieties with Picard number one of a given dimension, [21] and [23] have shown that birational boundedness fails. The problem is that the category of klt singularities is too big to be bounded since, for example, it contains finite quotients of arbitrarily large order. To get boundedness, one restricts to a smaller class of singularities, known as ϵ -klt singularities. Precisely we have the following conjecture due to A. Borisov, L. Borisov, and V. Alexeev, which is still open in dimension three and higher.

Borisov-Alexeev-Borisov Conjecture. Fix $0 < \epsilon < 1$, an integer $n > 0$, and consider the set of all n -dimensional ϵ -klt log \mathbb{Q} -Fano pairs (X, Δ) . The set of underlying varieties $\{X\}$ is bounded.

A. Borisov and L. Borisov establish the B-A-B Conjecture for toric varieties in [9]. V. Alexeev establishes the two dimensional B-A-B Conjecture in [2] with a simplified argument given in [3]. Our original motivation for studying the B-A-B Conjecture is that it is related to the conjectural termination of flips in the minimal model program. According to [8], the log minimal model program, the a.c.c.¹ for minimal log discrepancies, and the B-A-B Conjecture in dimension $\leq d$ implies termination of log flips in dimension $\leq d+1$ for effective pairs.

The following questions concerning log \mathbb{Q} -Fano pairs (X, Δ) are relevant to the B-A-B Conjecture:

- (i) The Cartier index of $K_X + \Delta$ of an n -dimensional ϵ -klt log \mathbb{Q} -Fano pair (X, Δ) is bounded from above by a fixed integer $r(n, \epsilon)$ depending only on $n = \dim X$ and ϵ ;
- (ii) The volume $\text{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n$ of an n -dimensional ϵ -klt log \mathbb{Q} -Fano pair (X, Δ) is bounded from above by a fixed integer $M(n, \epsilon)$ depending only on $n = \dim X$ and ϵ ;
- (iii) (**Batyrev Conjecture**) For given positive integers n and r , consider the set of all n -dimensional klt log \mathbb{Q} -Fano pairs (X, Δ) with $r(K_X + \Delta)$ a Cartier divisor. The set of underlying varieties $\{X\}$ is bounded.

It is clear that the B-A-B Conjecture follows from (i) and (iii). Note that recently C. Hacon, J. McKernan, and C. Xu have announced a proof of the Batyrev Conjecture (iii). In general it is very hard to establish (i). Ambro in [5] has proved (i) for toric singularities when the boundaries have standard coefficients $\{1 - \frac{1}{\ell} | \ell \in \mathbb{Z}_{\geq 1}\} \cup \{1\}$. A necessary condition for (i) to hold is that we need to restrict the coefficients of boundaries to be in a fixed d.c.c. set. A counterexample for the general statement is given by the set of pairs $(\mathbb{P}^1, \frac{1}{N}\{\text{pt}\})$ for $N \geq 1$.

For the convenience of the reader, we include a well-known argument (to the experts) establishing the B-A-B Conjecture via condition (i) and (ii) in the cases $\Delta = 0$ or $\rho(X) = 1$.

Proposition 0.3. *Suppose that $\Delta = 0$ or $\rho(X) = 1$, then the B-A-B Conjecture holds if both (i) and (ii) above are true.*

Proof. Suppose that $\Delta = 0$ and let X be any ϵ -klt \mathbb{Q} -Fano variety of dimension n . The following statements together imply the B-A-B conjecture in this case:

- (1) The divisor $N(-K_X)$ is a very ample line bundle for a fixed N depending only on n and ϵ ;
- (2) The set of Hilbert polynomials $\mathfrak{F} = \{P(t) = \chi(\mathcal{O}_X(-NK_X)^{\otimes t})\}$ associated to all n -dimensional ϵ -klt \mathbb{Q} -Fano varieties is finite.

Indeed, statements (1) and (2) imply that the set of n -dimensional ϵ -klt \mathbb{Q} -Fano varieties is contained in a finite union of Hilbert schemes $\coprod_{P(t) \in \mathfrak{F}} \mathcal{H}_{P(t)}$, where each $\mathcal{H}_{P(t)}$ is Noetherian.

From (i), there is an upper bound $r(n, \epsilon)$ of the Cartier index of K_X depending only on n and ϵ . It follows that rK_X is a line bundle for $r = r(n, \epsilon)$. By [13], $|-mrK_X|$ is base point free for any $m > 0$ divisible by a constant $N_1(n) > 0$ depending only on $n = \dim X$. Since $|-mrK_X|$ is ample and base point free for $m > 0$ sufficiently divisible, it defines a finite morphism. By [14, Theorem 5.9], the map induced by $|-lrK_X|$ is birational for any $l > 0$ divisible by a constant $N_2(n) > 0$ depending only on $n = \dim X$. Since a finite birational morphism of normal varieties is an isomorphism, it follows that there exists an

¹An a.c.c. (respectively d.c.c.) set is a set of real numbers satisfying the ascending (descending) chain condition, i.e., it contains no infinite strictly increasing (decreasing) sequences.

effective embedding by $|M(-rK_X)|$ for some fixed $M > 0$ depending only on $n = \dim X$. Take $N = Mr$, we have (1).

By [16], the coefficients of the Hilbert polynomial $P(t) = h^0(\mathcal{O}_X(tH))$ of a polarized variety (X, H) with H an ample line bundle can be bounded by the intersection numbers $|H^n|$ and $|H^{n-1} \cdot K_X|$. Since by (i) there exists an integer $r = r(n, \epsilon) > 0$ depending only on $n = \dim X$ and ϵ such that $-rK_X$ is an ample line bundle, set $H = -rK_X$ and apply (ii). It follows that there are only finitely many Hilbert polynomials for the set of anti-canonically polarized ϵ -klt Fano varieties $\{(X, -rK_X)\}$.

If $\rho(X) = 1$, then $-(K_X + \Delta)$ being ample implies that $-K_X$ is also ample. It is clear that X is also ϵ -klt and hence boundedness follows from the same proof as above. \square

An effective upper bound in (ii) is obtained for smooth Fano n -folds in [17] and for canonical \mathbb{Q} -Fano threefolds in [18]. In this paper, we obtain an effective answer to question (ii) in dimension two, i.e., for log del Pezzo surfaces.

Theorem A. (Theorem 4.3) *Let (X, Δ) be an ϵ -klt weak log del Pezzo surface. The volume $\text{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2$ satisfies*

$$(K_X + \Delta)^2 \leq \max\{64, \frac{8}{\epsilon} + 4\}.$$

Moreover, this upper bound is in a sharp form: There exists a sequence of ϵ -klt del Pezzo surfaces whose volume grows linearly with respect to $1/\epsilon$.

Let (X, Δ) be an ϵ -klt weak log del Pezzo surface and X_{\min} be the minimal resolution of (X, Δ) . Alexeev and Mori have shown in [3, Theorem 1.8] that $\rho(X_{\min}) \leq 128/\epsilon^5$. Also from [3, Lemma 1.2] (or see the proof of Theorem 4.3), an exceptional curve E on X_{\min} over X has degree $1 \leq -E^2 \leq 2/\epsilon$. When $\Delta = 0$, since the Cartier index of K_X is bounded from above by the determinant of the intersection matrix $(E_i \cdot E_j)$ of the exceptional curves E_i 's on X_{\min} over X , it follows that the Cartier index bound $r(2, \epsilon)$ in the statement (i) satisfies

$$(\diamond) \quad r(2, \epsilon) \leq 2(2/\epsilon)^{128/\epsilon^5}.$$

An upper bound of $(K_X + \Delta)^2$ is implicitly mentioned in [2] but not clearly written down. It is also not clear if the upper bound (\diamond) is optimal. In view of Theorem A, this seems unlikely.

As a second result, we also obtain an upper bound of the volumes for ϵ -klt \mathbb{Q} -factorial log \mathbb{Q} -Fano threefolds of Picard number one. Recall that a variety X is \mathbb{Q} -factorial if each Weil divisor is \mathbb{Q} -Cartier.

Theorem B. (Theorem 5.16) *Let (X, Δ) be an ϵ -klt \mathbb{Q} -factorial log \mathbb{Q} -Fano threefold of $\rho(X) = 1$. The degree $-K_X^3$ satisfies*

$$-K_X^3 \leq \left(\frac{24M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 12 \right)^3,$$

where $R(2, \epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$ and $M(2, \epsilon)$ is an upper bound of the volume $\text{Vol}(-K_S) = K_S^2$ for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$. Note that $M(2, \epsilon) \leq \max\{64, 16/\epsilon + 4\}$ from Theorem A and $R(2, \epsilon) \leq 2(4/\epsilon)^{128 \cdot 2^5/\epsilon^5}$ from (\diamond) .

For a \mathbb{Q} -factorial ϵ -klt log \mathbb{Q} -Fano pair (X, Δ) of $\rho(X) = 1$, since $-(K_X + \Delta)^3 \leq -K_X^3$ and X is also ϵ -klt, by Theorem B we get an upper bound of the anticanonical volume

$\text{Vol}(-(K_X + \Delta)) = -(K_X + \Delta)^3$. However, it is not expected that the bound in Theorem B is sharp or in a sharp form.

Note that \mathbb{Q} -factoriality is a technical assumption. However, this condition is natural in the sense that starting from a smooth variety, each variety constructed by a step of the minimal model program remains \mathbb{Q} -factorial. In dimension two, normal surfaces with rational singularities, e.g., klt singularities, are always \mathbb{Q} -factorial.

Instead of using deformation theory of rational curves as in [18], the Riemann-Roch formula as in [12], or the sandwich argument of [2], we aim to create isolated non-klt centers by the method developed in [22]. The point is that deformation theory for rational curves on klt varieties is much harder and so far no effective Riemann-Roch formula is known for klt threefolds.

The rest of this paper is organized as follows: In Section 1, we study non-klt centers. In Section 2, we illustrate the general method in [22] for obtaining an upper bound of anticanonical volumes in Theorem A and B. In Section 3, we review the theory of families of non-klt centers in [22]. In Section 4, we study weak log del Pezzo surfaces and prove Theorem A. In Section 5, we prove Theorem B.

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1. NON-KLT CENTERS

For the theory of the singularities in the minimal model program, we refer to [19].

Definition 1.1. Let (X, Δ) be a pair. A subvariety $V \subseteq X$ is called a **non-klt center** if it is the image of a divisor of discrepancy at most -1 . A **non-klt place** is a valuation corresponding to a divisor of discrepancy at most -1 . The **non-klt locus** $\text{Nklt}(X, \Delta) \subseteq X$ is the union of the non-klt centers. If there is a unique non-klt place lying over the generic point of a non-klt center V , then we say that V is **exceptional**. If (X, Δ) is not klt along the generic point of a non-klt center V , then we say that V is **pure**.

The non-klt places/centers here are the log canonical (lc) places/centers in [22].

A standard way of creating a non-klt center on an n -dimensional variety X is to find a very singular divisor: Fix $p \in X$ a smooth point, if Δ is a \mathbb{Q} -Cartier divisor on X with $\text{mult}_p \Delta \geq n$, then $p \in \text{Nklt}(X, \Delta)$. Indeed, consider the blow up $\pi : Y = \text{Bl}_p X \rightarrow X$ and let E be the unique exceptional divisor with $\pi(E) = p$, then the discrepancy

$$a(E, X, \Delta) = \text{mult}_E(K_Y - \pi^*(K_X + \Delta)) = (n - 1) - \text{mult}_E(\pi^*(\Delta)) \leq -1,$$

as $n - 1 = \text{mult}_E(K_Y - \pi^*K_X)$ and $\text{mult}_E(\pi^*\Delta) = \text{mult}_p \Delta \geq n$.

We can find singular divisors by the following lemma.

Lemma 1.2. Let X be an n -dimensional complete complex variety and D be a divisor with $h^i(X, \mathcal{O}(mD)) = O(m^{n-1})$ for all $i > 0$, e.g., D is big and nef. Fix a positive rational number α with $0 < \alpha^n < D^n$. For $m \gg 0$ and any $x \in X_{\text{sm}}$, there exists a divisor $E_x \in |mD|$ with $\text{mult}_x(E_x) \geq m \cdot \alpha$.

Proof. This is [20, Proposition 1.1.31]. □

We will apply Lemma 1.2 to the case where (X, Δ) is an n -dimensional log \mathbb{Q} -Fano pair: Write $-(K_X + \Delta)^n > (\omega n)^n$ for some rational number $\omega > 0$, then as $h^i(X, \mathcal{O}(-m(K_X + \Delta))) = 0$ for $m > 0$ sufficiently divisible by the Kawamata-Viehweg vanishing theorem, we can find for each $p \in X_{\text{sm}}$ a \mathbb{Q} -divisor Δ_p such that $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ and $\text{mult}_p(\Delta_p) \geq n$. In particular, $p \in \text{Nklt}(X, \Delta + \Delta_p)$.

The non-klt centers satisfy the following Connectedness Lemma of Kollár and Shokurov, which is simply a formal consequence of the Kawamata-Viehweg vanishing theorem and is the most important ingredient in this paper.

Lemma 1.3. *Let (X, Δ) be a log pair. Let $f : X \rightarrow Z$ be a projective morphism with connected fibers such that the image of every component of Δ with negative coefficient is of codimension at least two in Z . If $-(K_X + \Delta)$ is big and nef over Z , then the intersection of $\text{Nklt}(X, \Delta)$ with each fiber $X_z = f^{-1}(z)$ is connected.*

Proof. For simplicity, we assume that $Z = \text{Spec}(\mathbb{C})$ is a point and (X, Δ) is log smooth, i.e., X is smooth and Δ has simple normal crossing support. Then the identity map $\text{id}_X : X \rightarrow X$ is a log resolution of (X, Δ) and $\text{Nklt}(X, \Delta) = {}_{\perp}\Delta_{\perp}$. Consider the exact sequence

$$\cdots \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_{{}_{\perp}\Delta_{\perp}}) \rightarrow H^1(X, \mathcal{O}_X(-{}_{\perp}\Delta_{\perp})) \rightarrow \cdots.$$

Since $-{}_{\perp}\Delta_{\perp} = K_X + \{\Delta\} - (K_X + \Delta)$ and $(X, \{\Delta\})$ is klt, we have $H^1(X, \mathcal{O}_X(-{}_{\perp}\Delta_{\perp})) = 0$ by the Kawamata-Viehweg vanishing theorem as $-(K_X + \Delta)$ is nef and big. Since $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, we see that $\text{Nklt}(X, \Delta) = {}_{\perp}\Delta_{\perp}$ is connected.

For the general case, see [10, Theorem 17.4]. \square

Here is an example showing that $-(K_X + \Delta)$ being nef and big is necessary in the Connectedness Lemma 1.3.

Example 1.4. Let X be $\mathbb{P}^1 \times \mathbb{P}^1$ and denote by F (resp. G) the fiber of the first (resp. second) projection to \mathbb{P}^1 . Consider $\Delta_1 = F_1 + F_2$ the sum of two distinct fibers of the first projection to \mathbb{P}^1 and $\Delta_2 = F + G$ the sum of two fibers with respect to the two different projections to \mathbb{P}^1 . Then $\text{Nklt}(X, \Delta_1) = F_1 + F_2$ is not connected while $\text{Nklt}(X, \Delta_2) = F + G$ is connected. Note that $-(K_X + \Delta_1)$ is nef but not big while $-(K_X + \Delta_2)$ is nef and big.

Later on, we will produce not only non-klt centers but *isolated* non-klt centers. The following theorem is the main technique which allows us to cut down the dimension of non-klt centers.

Theorem 1.5. ([14, Theorem 6.8.1]) *Let (X, Δ) be klt, projective and $x \in X$ a closed point. Let D be an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $(X, \Delta + D)$ is log canonical in a neighborhood of x . Assume that $\text{Nklt}(X, \Delta + D) = Z \cup Z'$ where Z is irreducible, $x \in Z$, and $x \notin Z'$. Set $k = \dim Z$. If H is an ample \mathbb{Q} -divisor on X such that $(H^k \cdot Z) > k^k$, then there is an effective \mathbb{Q} -divisor $B \equiv H$ and rational numbers $1 \gg \delta > 0$ and $0 < c < 1$ such that*

- (1) $(X, \Delta + (1 - \delta)D + cB)$ is non-klt in a neighborhood of x , and
- (2) $\text{Nklt}(X, \Delta + (1 - \delta)D + cB) = W \cup W'$ where W is irreducible, $x \in W$, $x \notin W'$ and $\dim W < \dim Z$.

2. A GUIDING EXAMPLE

The idea in [22] for obtaining an upper bound for the anticanonical volumes is to create isolated non-klt centers and then use the Connectedness Lemma 1.3: For simplicity, we assume that $\Delta = 0$. Write $(-K_X)^n > (\omega n)^n$ for a positive rational number ω . For each $p \in X_{\text{sm}}$, we can find an effective \mathbb{Q} -divisor $\Delta_p \sim_{\mathbb{Q}} -K_X/\omega$ such that $\text{mult}_p \Delta_p \geq n$ and hence $p \in \text{Nklt}(X, \Delta_p)$. The observation is that if $\omega \gg 0$, then for general $p \in X$, $p \in \text{Nklt}(X, \Delta_p)$ can not be an isolated point. Indeed, if this is not true, then for two general points $p, q \in X$, the set $\text{Nklt}(X, \Delta_p + \Delta_q)$ would contain $\{p, q\}$ as isolated non-klt centers. But the divisor $K_X + \Delta_p + \Delta_q \sim_{\mathbb{Q}} (1 - \frac{2}{\omega})(-K_X)$ is nef and big for $\omega > 2$. By the Connectedness Lemma 1.3, $\text{Nklt}(X, \Delta_p + \Delta_q)$ must be connected; a contradiction.

Therefore, for general $p \in X$ the minimal non-klt center $V_p \subseteq \text{Nklt}(X, \Delta_p)$ passing through p is typically positive dimensional. We would like to show that the restricted volume $\text{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p is large when $\omega \gg 0$. Hence, we can cut down the dimension

of non-klt centers by Theorem 1.5. After doing this finitely many times, we get isolated non-klt centers and we are done.

In general, it is hard to find a lower bound of the restricted volume $\text{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p . We illustrate McKernan's method by studying families of non-klt centers to obtain a lower bound of the restricted volumes on the non-klt center of an ϵ -klt log \mathbb{Q} -Fano variety via the following guiding example, cf. [22].

Example 2.1. Let X be the projective cone over a rational normal curve of degree $d \geq 2$ with the unique singular point $O \in X$. The blow up $\pi : Y = \text{Bl}_O X \rightarrow X$ is a resolution of X where Y is a \mathbb{P}^1 -bundle $f : Y \rightarrow \mathbb{P}^1$ over \mathbb{P}^1 :

$$\begin{array}{ccc} X & \xleftarrow{\pi} & Y \supseteq F_t \cong \mathbb{P}^1 \\ & & \downarrow f \\ & & \mathbb{P}^1 \ni t. \end{array}$$

It is easy to show that

- (a) $K_Y = \pi^*K_X + (-1 + 2/d)E$, where E is the unique exceptional divisor and hence X is ϵ -klt for $\epsilon = 1/d$;
- (b) X is \mathbb{Q} -factorial of Picard number one and $-K_X \sim_{\mathbb{Q}} (d+2)l$ is an ample \mathbb{Q} -Cartier divisor, where l is the class of a ruling of X . Hence X is an ϵ -klt del Pezzo surface;
- (c) $\text{Vol}(-K_X) = d + 4 + 4/d$ is a linear function of $d = 1/\epsilon$ and provides the required example in Theorem A.

Let $p \in X$ be a general point. Then p is not the vertex O and the unique ruling l_p passing through p is the non-klt center of the log pair (X, l_p) , i.e., $l_p = \text{Nklt}(X, l_p)$. Moreover, the proper transform F_p of l_p on Y is a fiber of the \mathbb{P}^1 -bundle $f : Y \rightarrow \mathbb{P}^1$. In this case, the \mathbb{P}^1 -bundle structure of Y is a covering family of non-klt centers of X since the map $\pi : Y \rightarrow X$ is dominant.

For $p, q \in X$ two general points, let l_p and l_q be the rulings passing through p and q respectively. Consider the pair $K_Y + (1 - 2/d)E = \pi^*K_X$. By the Connectedness Lemma 1.3, the non-klt locus $\text{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q))$ containing $F_p \cup F_q$ is connected as

$$-(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q)) = -\pi^*(K_X + l_p + l_q) \equiv d\pi^*l$$

is nef and big. In fact, the fibers F_p and F_q are connected in $\text{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q))$ by E as

$$F_p \cup F_q \subseteq \text{Nklt}(K_Y + (1 - 2/d)E + \pi^*(l_p + l_q)) \subseteq \pi^{-1}(\text{Nklt}(K_X + l_p + l_q)) = F_p \cup F_q \cup E,$$

where the second inclusion follows from the definition of non-klt centers. In particular,

$$\text{mult}_E(\pi^*(l_p + l_q)) \geq \frac{2}{d} = 2\epsilon.$$

By symmetry, π^*l_p must contribute multiplicity at least $1/d = \epsilon$ to the component E (and in fact is exactly $1/d$ in this case), i.e.,

$$(2.1) \quad \pi^*l_p \geq \epsilon E.$$

Note that

$$(2.2) \quad l_p \sim_{\mathbb{Q}} \frac{-K_X}{\sqrt{d} \cdot \text{Vol}(-K_X)}.$$

By intersecting both sides of (2.1) with a general fiber F of $f : Y \rightarrow \mathbb{P}^1$, we get for the ruling $l = \pi_*(F)$,

$$(2.3) \quad \frac{1}{\sqrt{d \cdot \text{Vol}(-K_X)}} \deg_l(-K_X) = \pi^* l_p \cdot F \geq \epsilon E \cdot F.$$

Since F is a general fiber meeting the horizontal divisor E at a smooth point, $E \cdot F \geq 1$. (In this case $E \cdot F = 1$.) Combining all of these, we obtain a lower bound of the restricted volume $\deg_l(-K_X)$,

$$\deg_l(-K_X) \geq \epsilon \sqrt{d \cdot \text{Vol}(-K_X)}.$$

Note that since in this case $\deg_l(-K_X) = -K_X \cdot l = -K_Y \cdot \pi^* l \leq 2$, it follows that $\text{Vol}(-K_X) = K_X^2 \leq 4d = 4/\epsilon$.

In summary, the method of getting an upper bound of the anticanonical volumes is to obtain a lower bound of the restricted volume $\text{Vol}(-(K_X + \Delta)|_{V_p})$ on the non-klt centers V_p , which can be outlined in the following steps:

- Suppose that $\text{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n > (\omega n)^n$ for a positive rational number ω . We will show that $\omega > 0$ can not be arbitrarily large.
- For general $p \in X$, choose

$$\Delta_p \sim_{\mathbb{Q}} \frac{-(K_X + \Delta)}{\omega},$$

so that $p \in \text{Nklt}(X, \Delta + \Delta_p)$. Let $V_p \subseteq \text{Nklt}(X, \Delta + \Delta_p)$ be the minimal non-klt center containing p .

- Construct covering families of non-klt centers by “lining up” (part of the) non-klt centers $\{V_p\}$, see Section 3. This is the generalization of the \mathbb{P}^1 -bundle structure in the Example 2.1 and is called a *covering families of tigers* in [22].
- Use the Connectedness Lemma 1.3 to obtain a lower bound of the restricted volume

$$\text{Vol}(-(K_X + \Delta)|_{V_p}) = (-(K_X + \Delta)|_{V_p})^{\dim V_p},$$

on the non-klt center V_p in terms of ω and ϵ . This is the most technical part.

- If $\omega \gg 0$, then we cut down the dimension of non-klt centers by Theorem 1.5. After finitely many steps, we get isolated non-klt centers and hence a contradiction to the Connectedness Lemma 1.3.

The difficulty of this argument arises in dimension three in many places. First of all, the non-klt centers can be of dimension one or two and we have to deal with them case by case. When we have one dimensional covering families of tigers, it is subtle to detect the contribution of the ϵ -klt condition from some horizontal subvariety, which is analogous to the exceptional curve E in Example 2.1. This is done by applying a differentiation argument to construct a better behaved covering family of tigers, see 5.3. In case we have two dimensional non-klt centers, complications arise for computing intersection numbers as the total space Y of a covering family of tigers is in general not \mathbb{Q} -factorial. This can be fixed by replacing Y with a suitable birational model. To finish the proof, we also need to run a relative minimal model on the covering family of tigers and study the geometry of all possible outcomes.

3. COVERING FAMILIES OF TIGERS

The main reference for this section is [22].

Definition 3.1. ([22, Definition 3.1]) *Let (X, Δ) be a log pair with X projective and D a \mathbb{Q} -Cartier divisor. We say that pairs of the form (Δ_t, V_t) form a **covering family of tigers** of dimension k and weight ω if all of the following hold:*

- (1) there is a projective morphism $f : Y \rightarrow B$ of normal projective varieties such that the general fiber of f over $t \in B$ is V_t ;
- (2) there is a morphism of B to the Hilbert scheme of X such that B is the normalization of its image and f is obtained by taking the normalization of the universal family;
- (3) if $\pi : Y \rightarrow X$ is the natural morphism, then $\pi(V_t)$ is a minimal pure non-klt center of $K_X + \Delta + \Delta_t$;
- (4) π is generically finite and dominant;
- (5) $\Delta_t \sim_{\mathbb{Q}} D/\omega$, where Δ_t is effective;
- (6) the dimension of V_t is k .

Note that by definition $k \leq \dim X - 1$ and $\pi|_{V_t} : V_t \rightarrow \pi(V_t)$ is finite and birational. The covering family of tigers is illustrated in the following diagram:

$$\begin{array}{ccc}
 X & \xleftarrow{\pi} & Y \supseteq V_t \\
 & & \downarrow f \\
 & & B \ni t.
 \end{array}$$

We will sometimes also refer to V_t as the minimal non-klt center of $(X, \Delta + \Delta_t)$.

For (X, Δ) a log \mathbb{Q} -Fano variety, we will always assume that $D = -\lambda(K_X + \Delta)$ for some $\lambda > 0$. In particular, D is assumed to be big and semi-ample.

The existence of a covering family of tigers is achieved by constructing non-klt centers at general points of X and then fitting a sub-collection of them into a fiber space. In order to fit the non-klt centers into a family, we use exceptional non-klt centers so that we patch up the unique non-klt place associated to each of them. The following lemma allows us to create exceptional non-klt centers.

Lemma 3.2. *Let (X, Δ) be a log pair and let D be a big and semi-ample \mathbb{Q} -Cartier divisor. Write $D^n > (\omega n)^n$ for some positive rational number ω . In order to find an upper bound of ω and hence an upper bound of $\text{Vol}(D) = D^n$, for every $p \in X_{\text{sm}}$ we may assume that there is a divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ such that the unique minimal non-klt center $V_p \subseteq \text{Nklt}(X, \Delta + \Delta_p)$ containing p is exceptional.*

Proof. By Lemma 1.2, for any $p \in X_{\text{sm}}$ we can find an effective divisor $\Delta'_p \sim_{\mathbb{Q}} \frac{D}{\omega}$ such that $\text{mult}_p \Delta'_p \geq n$ and hence $p \in \text{Nklt}(X, \Delta + \Delta'_p)$.

Fix $p \in X_{\text{sm}}$, pick $0 < \delta_p \leq 1$ the unique rational number such that $(X, \Delta + \delta_p \Delta'_p)$ is log canonical but not klt at p . By [4, Proposition 3.2, Lemma 3.4], we can find an effective divisor $M_p \sim_{\mathbb{Q}} D$ and some rational number $a > 0$ such that for any rational number $0 < \mu < 1$, the pair $(X, (1 - \mu)(\Delta + \delta_p \Delta'_p) + \mu\Delta + \mu a M_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. If we write

$$\Delta_p := (1 - \mu)\delta_p \Delta'_p + \mu a M_p \sim_{\mathbb{Q}} \frac{1}{\omega'_p} D,$$

then

$$\omega'_p = \frac{\omega}{(1 - \mu)\delta_p + \mu a \omega},$$

and $(1 - \mu)\delta_p + \mu a \omega < 1 + 1/n$ for any $n \geq 1$ if we pick $0 < \mu \ll 1$ sufficiently small. Hence $\omega'_p > \omega/(1 + 1/n)$. Since D is semi-ample, by adding a small multiple of D to Δ_p we have $\Delta_p \sim_{\mathbb{Q}} D/\omega_n$ for $\omega_n = \omega/(1 + 2/n)$, and $(X, \Delta + \Delta_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. If there exists an upper bound of ω_n independent of n , then by taking $n \rightarrow \infty$, we get the same upper bound of ω . \square

The following proposition is the construction of the covering family of tigers, see [22, Lemma 3.2] or [24, Lemma 3.2].

Proposition 3.3. *Let (X, Δ) and Δ_p be the same as in Lemma 3.2. Then there exists a covering family of tigers $\pi : Y \rightarrow X$ of weight ω with $V_p \subseteq \text{Nklt}(X, \Delta + \Delta_p)$ the unique minimal non-klt center passing through p .*

Proof. Choose $m > 0$ an integer such that mD/ω is integral and Cartier and let B be the Zariski closure of points $\{m\Delta_p | p \in X_{\text{sm}}\} \in |mD/\omega|$. Replace B by an irreducible component which contains an uncountable subset Q of B such that the set $\{p \in X | \Delta_p \in Q\}$ is dense in X . This is possible since the Δ_p 's cover X . Let $H \subseteq X \times |mD/\omega|$ be the universal family of divisors defined by the incidence relation and $H_B \rightarrow B$ the restriction to B . Take a log resolution of $H_B \subseteq X \times B$ over the generic point of B and extend it over an open subset U of B . By assumption the log resolution over the generic point of B has a unique exceptional divisor of discrepancy -1 , since this is true over $Q \subseteq B$. Let Y be the image of this unique exceptional divisor in $X \times B$ with the natural projection map $\pi : Y \rightarrow X$. By construction $\pi : Y \rightarrow X$ dominates X .

Possibly taking a finite cover of B and passing to an open subset of B , we may assume that any fiber V_t of $f : Y \rightarrow B$ over $t \in B$ is a non-klt center of $K_X + \Delta + \Delta_t$. Possibly passing to an open subset of B , we may assume that $f : Y \rightarrow B$ is flat and B maps into the Hilbert scheme. Replace B by the normalization of the closure of its image in the Hilbert scheme and Y by the normalization of the pullback of the universal family. After possibly cutting by hyperplanes in B , we may assume that π is generically finite and dominant. The resulting family is the required covering family of tigers. \square

In fact, the original construction of covering families of tigers is carried out in a more general setting. For a topological space X , we say that a subset P is *countably dense* if P is not contained in the union of countable many closed subsets of X .

Corollary 3.4. *Let (X, Δ) be a log pair and let D be a big \mathbb{Q} -Cartier divisor. Let ω be a positive rational number. Let P be a countably dense subset of X . If for every point $p \in P$ we may find a pair (Δ_p, V_p) such that V_p is a pure non-klt center of $K_X + \Delta + \Delta_p$, where $\Delta_p \sim_{\mathbb{Q}} D/\omega_p$ for some $\omega_p > \omega$, then we may find a covering family of tigers of weight ω together with a countably dense subset Q of P such that for all $q \in Q$, V_q is a fiber of π .*

Proof. See [22, Lemma 3.2] or [24, Lemma 3.2]. \square

As noted in Example 2.1, we can assume that the covering families of tigers under our consideration are always positive dimensional.

Lemma 3.5. *Let (X, Δ) be a projective klt pair and $D = -(K_X + \Delta)$ be a big and nef \mathbb{Q} -Cartier divisor. A covering family of tigers (Δ_t, V_t) of weight $\omega > 2$ is positive dimensional, i.e., $k = \dim V_t > 0$.*

Proof. This is [22, Lemma 3.4] and we include the proof for the convenience of the reader. Suppose that there exists a zero dimensional covering family of tigers of weight $\omega > 2$. For p_1 and p_2 general, there are divisors Δ_1 and Δ_2 with $\Delta_i \sim_{\mathbb{Q}} D/\omega$ such that p_i is an isolated non-klt center of $K_X + \Delta + \Delta_i$. As p_1 and p_2 are general, it follows that Δ_2 does not contain p_1 and $\text{Nklt}(X, \Delta + \Delta_1 + \Delta_2)$ contains p_1 and p_2 as disconnected non-klt centers. But $-(K_X + \Delta + \Delta_1 + \Delta_2) \sim (1 - \frac{2}{\omega})D$ is nef and big if $\omega > 2$. This contradicts Lemma 1.3. \square

Recall that we want to cut down the dimension of non-klt centers via Theorem 1.5. To do so, we study the associated covering families of tigers and obtain a lower bound of restricted volumes on the non-klt centers. If the new non-klt centers after cutting down the dimension are still positive dimensional, then we have to create new covering families of tigers associated to these new non-klt centers and repeat the process. The following proposition enables us to create covering families of tigers of new non-klt centers after cutting down the dimension.

Proposition 3.6. *Let (X, Δ) be a log pair and let D be a \mathbb{Q} -Cartier divisor of the form $A + E$ where A is ample and E is effective. Let (Δ_t, V_t) be a covering family of tigers of weight ω and dimension k . Let A_t be $A|_{V_t}$. If there is an open subset $U \subseteq B$ such that for all $t \in U$ we may find a covering family of tigers $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight ω' with respect to A_t , then for (X, Δ) we can find a covering family of tigers (Γ_s, W_s) of dimension less than k and weight*

$$\omega'' = \frac{1}{1/\omega + 1/\omega'} = \frac{\omega\omega'}{\omega + \omega'}.$$

Proof. This is [22, Lemma 5.3]. □

We will apply Proposition 3.6 with the ample divisor $D = -(K_X + \Delta)$. In the process of obtaining lower bound of the restricted volume on the non-klt centers, if we have one-dimensional non-klt centers, then we can control the restricted volume of D , cf. [22, Lemma 5.3].

Corollary 3.7. *Let (X, Δ) be a log pair and let D be an ample divisor. Let (Δ_t, V_t) be a covering family of tigers of weight $\omega > 2$ and dimension one. Then $\deg(D|_{V_t}) \leq 2\omega/(\omega - 2)$.*

Proof. Suppose that $\deg(D|_{V_t}) > 2\omega/(\omega - 2)$. By Lemma 3.2 and Corollary 3.4, we may find a covering family $(\Gamma_{t,s}, W_{s,t})$ of tigers of weight $\omega' > 2\omega/(\omega - 2)$ and dimension zero on V_t . By Proposition 3.6, there exists a covering family of tigers of dimension zero and weight

$$\omega'' = \frac{\omega\omega'}{\omega + \omega'} > 2,$$

for X . This contradicts Lemma 3.5. □

4. LOG DEL PEZZO SURFACES

Let (X, Δ) be an ϵ -klt weak log del Pezzo surface. The minimal resolution $\pi : Y \rightarrow X$ of (X, Δ) is the unique proper birational morphism such that Y is a smooth projective surface and $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ for some effective \mathbb{Q} -divisor Δ_Y on Y . Note that minimal resolutions always exist for two-dimensional log pairs. It is easy to see that (Y, Δ_Y) is also an ϵ -klt weak log del Pezzo surface with volume

$$\text{Vol}(Y, \Delta_Y) = (K_Y + \Delta_Y)^2 = (K_X + \Delta_X)^2 = \text{Vol}(X, \Delta_X).$$

Replacing (X, Δ) by its minimal resolution, we can assume that X is smooth.

Write $(K_X + \Delta)^2 > (2\omega)^2$. For a general point $p \in X$, let $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ be an effective \mathbb{Q} -divisor constructed from Lemma 1.2 such that $p \in \text{Nklt}(X, \Delta + \Delta_p)$. Assume that $\omega > 2$. By Lemma 3.5, the unique minimal non-klt center F_p of $(X, \Delta + \Delta_p)$ containing p is one dimensional. Note that for general $p \in X$, $F_p \leq \Delta_p$.

Lemma 4.1. *For a very general point $p \in X$, the numerical class $F := F_p$ on X is well-defined and F is nef.*

Proof. The effective integral one cycles F_p satisfy $F_p \leq \Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ and hence form a bounded set in the Mori cone of curves. As \mathbb{C} is uncountable, for $p \in X$ a very general point the numerical class $F := F_p$ is well-defined. Since $\{F_p\}$ moves, the class F is nef. □

The following lemma shows that if we assume the weight ω is large, then the non-klt centers $\{F_p\}$ on X already possess a nearly fiber bundle structure analogous to a covering family of tigers.

Lemma 4.2. *Assume that $\omega > 3$, then $F^2 = 0$, i.e. $F_p \cap F_q = \emptyset$ for $p, q \in X$ two very general points.*

Proof. Assume that $F_p \cap F_q \neq \emptyset$ for $p, q \in X$ two very general points. We can assume that $p \notin \Delta_q$ as $p \in X$ is very general. Since by Lemma 4.2 the curve class $F = F_p$ is nef, for $H = -(K_X + \Delta)/\omega$ we have

$$1 \leq F_p \cdot F_q = F_p \cdot F \leq \Delta_p \cdot F = \deg(H|_{F_p}),$$

where the first inequality is true since X is smooth. Since H is big and nef, we can cut down the dimension of the non-klt centers by Theorem 1.5².

To be precise, pick $0 < \delta_1 \leq 1$ such that the pair $(X, \Delta + \delta_1 \Delta_p)$ is log canonical but not klt at p . If $(X, \Delta + \delta_1 \Delta_p) = \{p\}$, then this contradicts the Connected Lemma 1.3 as $p \notin \Delta_q$ and the non-klt locus $\text{Nklt}(X, \Delta + \delta_1 \Delta_p + \Delta_q)$ containing p and F_q is disconnected, while the divisor $-(K_X + \Delta + \delta_1 \Delta_p + \Delta_q)$ is nef and big. Hence we may assume that $\text{Nklt}(X, \Delta + \delta_1 \Delta_p)$ is one dimensional in a neighborhood of p . In particular, $F_p \subseteq \text{Nklt}(X, \Delta + \delta_1 \Delta_p)$ is the minimal non-klt center containing p . By Theorem 1.5, there exists rational numbers $0 < \delta \ll 1$, $0 < c < 1$, and an effective \mathbb{Q} -divisor $B_p \equiv H$ such that $\text{Nklt}(X, \Delta + (1 - \delta)\delta_1 \Delta_p + cB_p) = \{p\}$ in a neighborhood of p . It follows that the set of non-klt centers $\text{Nklt}(X, \Delta + (1 - \delta)\delta_1 \Delta_p + cB_p + \Delta_q)$ containing p and F_q is disconnected but the divisor $-(K_X + \Delta + (1 - \delta)\delta_1 \Delta_p + cB_p + \Delta_q)$ is nef and big as $\omega > 3$. This again contradicts the Connected Lemma 1.3. \square

Theorem 4.3. *Let (X, Δ) be an ϵ -klt weak log del Pezzo surface. Then the anticanonical volume $\text{Vol}((-K_X + \Delta)) = (K_X + \Delta)^2$ satisfies*

$$(K_X + \Delta)^2 \leq \max\{64, \frac{8}{\epsilon} + 4\}.$$

Proof. Replacing (X, Δ) by its minimal resolution, we may assume that X is smooth. Write $(K_X + \Delta)^2 > (2\omega)^2$. For each general point $p \in X$, by Lemma 1.2, there exists an effective \mathbb{Q} -divisor $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ such that $p \in \text{Nklt}(X, \Delta + \Delta_p)$. From Lemma 3.5, we may assume that $\omega > 2$ and the unique minimal non-klt center $F_p \subseteq \text{Nklt}(X, \Delta + \Delta_p)$ containing p is one dimensional. Note that $F_p \leq \Delta_p$ for general $p \in X$. By Lemma 4.1 and 4.2, we may assume that $\omega > 3$ and for very general $p \in X$ the numerical class F of F_p is well-defined and nef with $F^2 = 0$.

For two very general points $p, q \in X$, $\Delta_p \cdot \Delta_q > 0$ and hence $F_p = \text{Supp}(F_p) \subsetneq \text{Supp}(\Delta_p)$: Otherwise $\Delta_q \equiv \Delta_p \leq N F_p$ for some $N > 0$ and $0 < \Delta_p \cdot \Delta_q \leq N^2 F_p^2 = N^2 F^2 = 0$, a contradiction. By the Connectedness Lemma 1.3, $\text{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q$ is connected. Denote $E_p = \text{Supp}(\Delta_p) - F_p \neq \emptyset$. By Lemma 4.2, $F_p \cap F_q = \emptyset$ and hence E_p must contain a connected curve $E \leq E_p$ such that $F_p \cdot E \neq 0$, $F_q \cdot E \neq 0$, and the set $\text{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q \cup E$. Furthermore, we can assume that E is irreducible since $E \cdot F_q \neq 0$ as $F_q \equiv F_p$ for $q \in X$ a very general point.

Suppose that $E^2 \geq 0$ and hence E is nef. Since $\text{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q \cup E$, we have $\Delta + \Delta_p + \Delta_q \geq E$ and $(\Delta + \Delta_p + \Delta_q - E) \cdot E \geq 0$. For $H = -(K_X + \Delta)/\omega$, we see that

$$\begin{aligned} 2 &\geq 2 - 2g_a(E) \geq -(K_X + E) \cdot E - (\Delta + \Delta_p + \Delta_q - E) \cdot E \\ &= -(K_X + \Delta + \Delta_p + \Delta_q) \cdot E \\ &= (\omega - 2)H \cdot E. \end{aligned}$$

Write $\Delta_p = \Delta'_p + \alpha E$ where $\Delta'_p \wedge E = 0$, $\Delta'_p \geq F_p$, and $\alpha > 0$, we have

$$H \cdot E = \Delta_p \cdot E = (\Delta'_p + \alpha E) \cdot E \geq F_p \cdot E \geq 1.$$

The last inequality follows from the fact that X is smooth and $F_p \cdot E > 0$. Combine the two inequalities above, we obtain $\omega \leq 4$.

²By adding a small multiple of $-(K_X + \Delta)$, we may assume that the inequality $\deg(H|_{F_q}) \geq 1$ is strict with a smaller modified ω and hence Theorem 1.5 applies.

Hence we may assume that $E^2 < 0$, and thus

$$\begin{aligned} -2 &\leq 2g_a(E) - 2 = (K_X + E).E \\ &= (K_X + \Delta).E + (1 - \epsilon - a_E)E^2 - \Delta'.E + \epsilon E^2 \leq \epsilon E^2, \end{aligned}$$

where $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [0, 1 - \epsilon)$ by the ϵ -klt condition. This implies that $1 \leq -E^2 \leq 2/\epsilon$, where the first inequality follows from the fact that $E^2 \in \mathbb{Z}$ as X is smooth. Since $F^2 = 0$ for F the numerical class of F_p where $p \in X$ is very general, by Nakai's criterion the divisor $H_s = F + sE$ with $0 < s \leq 1/(-E^2)$ is nef and big. By the Hodge index theorem (see [11, V 1.1.9(a)]), we get the inequality

$$(4.1) \quad (K_X + \Delta)^2 \leq \frac{(-(K_X + \Delta).H_s)^2}{H_s^2}.$$

From $\Delta.F \geq 0$ and $F^2 = 0$, we have that

$$(4.2) \quad -(K_X + \Delta).F \leq -(K_X + F).F \leq 2.$$

Also for $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [0, 1 - \epsilon)$, we have that

$$\begin{aligned} -(K_X + \Delta).E &= -K_X.E - \Delta'.E - a_E E^2 \\ (4.3) \quad &\leq E^2 + 2 - a_E E^2 = (a_E - 1)(-E^2) + 2 \leq 2 - \epsilon(-E^2). \end{aligned}$$

Put $s = 1/(-E^2)$, all together we get

$$\begin{aligned} (K_X + \Delta)^2 &\leq \frac{(-(K_X + \Delta).(F + sE))^2}{H_s^2} \\ &\leq \frac{(2 + s(2 - \epsilon(-E^2)))^2}{2sE.F + s^2 E^2} \\ &\leq (-E^2)(2 - \epsilon + \frac{2}{-E^2})^2 \\ &= (-E^2)(2 - \epsilon)^2 + 4(2 - \epsilon) + \frac{4}{-E^2} \\ &\leq \frac{2}{\epsilon}(2 - \epsilon)^2 + 4(2 - \epsilon) + 4 \\ &= \frac{8}{\epsilon} + 4 - 2\epsilon \end{aligned}$$

where the first inequality is (4.1), the second inequality follows from (4.2), (4.3), and $F^2 = 0$, the third inequality is given by ignoring the term $sE.F \geq 0$, and the last inequality uses $1 \leq -E^2 \leq 2/\epsilon$. \square

Remark 4.4. Note that by applying Corollary 3.7 one can only obtain an upper bound of order $1/\epsilon^2$. Hence Theorem 4.3 is a non-trivial result.

5. LOG FANO THREEFOLDS OF PICARD NUMBER ONE

Let (X, Δ) be an ϵ -klt \mathbb{Q} -factorial log \mathbb{Q} -Fano threefold of Picard number $\rho(X) = 1$. Note that by hypothesis X is ϵ -klt and $-K_X$ is ample with $-K_X^3 \geq \text{Vol}(-(K_X + \Delta)) = -(K_X + \Delta)^3$. Hence it is sufficient to assume that X is an ϵ -klt \mathbb{Q} -factorial \mathbb{Q} -Fano threefold of Picard number $\rho(X) = 1$ and to find an upper bound of $\text{Vol}(-K_X) = -K_X^3$. We will obtain an upper bound of the anticanonical volumes by studying covering families of tigers. The weight of any covering families of tigers in our study will always be the weight with respect to $-K_X$.

Let X be an ϵ -klt \mathbb{Q} -factorial \mathbb{Q} -Fano threefold of Picard number $\rho(X) = 1$ and write the anticanonical volume $\text{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω . Denote $D = -2K_X$, we have $D^3 > (6\omega)^3$. By Lemma 1.2, we can fix an affine open subset $U \subseteq X$ such

that for each $p \in U$ there exists an effective divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ with $\text{mult}_p \Delta_p \geq 6$. We pick divisors Δ_p 's in the following systematic way so that we can control their multiplicities uniformly.

5.1. Construction. Let $\Delta_U \subseteq U \times U$ be the diagonal and $\mathcal{I}_{\mathcal{Z}}$ be the ideal sheaf of the subvariety $\mathcal{Z} = \overline{\Delta_U} \subseteq X \times U$. For each $p \in U$, by the existence of \mathbb{Q} -divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ with $\text{mult}_p \Delta_p \geq 6$, there exists $m_p > 0$ such that $L_{m_p} = m_p D/\omega$ is Cartier and $H^0(X, L_{m_p} \otimes \mathcal{I}_p^{\otimes 6m_p}) \neq 0$. In particular, we can write $U = \bigcup U_m$ where $m > 0$ runs through all sufficiently divisible integers such that $L_m = mD/\omega$ is Cartier and $U_m = \{p \in U \mid H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}) \neq 0\}$. Moreover, each U_m is locally closed in X by [11, III, Theorem 12.8] and $X = \bigcup \overline{U_m}$. Since the base field \mathbb{C} is uncountable, X can not be a countable union of locally closed subsets. Thus there exists some $m > 0$ such that U_m is dense in X .

Fix an $m > 0$ such that $L_m = mD/\omega$ is Cartier and $U_m = \{p \in U \mid H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}) \neq 0\}$ is dense in X . Denote $\text{pr}_X : X \times U \rightarrow X$ and $\text{pr}_U : X \times U \rightarrow U$ the projection maps. Since $\text{pr}_U : X \times U \rightarrow U$ is flat, by [11, III, Theorem 12.11], after restricting to a smaller open affine subset of U , we can assume that the map

$$(\text{pr}_U)_*(\text{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \otimes \mathbb{C}(p) \rightarrow H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}),$$

is an isomorphism for each $p \in U$ where \mathcal{I}_p is the ideal sheaf of $p \in U$. Since U_m is dense in U , the sheaf $(\text{pr}_U)_*(\text{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ on U and hence $H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ as U is affine. Let $s \in H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with $F = \text{div}(s)$ the corresponding divisor on $X \times U$. For each $p \in U$, denote $F_p = F \cap (X \times \{p\})$ the associated divisor on $X \cong X \times \{p\}$. Since $\text{mult}_{\mathcal{Z}}(F) \geq 6m$, by Lemma 5.1 below, the \mathbb{Q} -divisor $\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega$ on X satisfies $\text{mult}_p \Delta_p \geq 6$ for general $p \in U$.

Lemma 5.1. ([20, Lemma 5.2.11]) *Let $g : M \rightarrow T$ be a morphism of smooth varieties, and suppose that $\mathcal{Z} \subseteq M$ is an irreducible subvariety dominating T :*

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\quad} & M \\ & \searrow & \swarrow g \\ & T. & \end{array}$$

Let $F \subseteq M$ be an effective divisor. For a general point $t \in T$ and an irreducible component $\mathcal{Z}'_t \subseteq \mathcal{Z}_t$, $\text{mult}_{\mathcal{Z}'_t}(M_t, F_t) = \text{mult}_{\mathcal{Z}}(M, F)$, where $\text{mult}_{\mathcal{Z}}(M, F)$ is the multiplicity of the divisor F on M along a general point of the irreducible subvariety $\mathcal{Z} \subseteq M$ and similarly for $\text{mult}_{\mathcal{Z}'_t}(M_t, F_t)$.

For a given collection of \mathbb{Q} -divisors $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega \mid p \in U \text{ general}\}$ associated to a nonzero section in $H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ as above, by Lemma 3.2, we can modify the Δ_p 's so that the unique non-klt centers $V_p \subseteq \text{Nklt}(X, \Delta_p)$ passing through p are exceptional. By Lemma 3.3 (or in general Corollary 3.4), we can construct covering families of tigers from these divisors.

In order to obtain an upper bound of ω , which is sufficient for bounding the anticanonical volumes, we will pick up a “well-behaved” nonzero section $s \in H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ and study the corresponding covering families of tigers.

5.2. Cases. By Section 5.1, there exists an open affine subset $U \subseteq X$ and an integer $m > 0$ such that $H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. Let $s \in H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with divisor $F = \text{div}(s)$ on $X \times U$ and $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega \mid p \in U\}$ be the associated collection of \mathbb{Q} -divisors. We consider two cases:

- (1) **(Small multiplicity)** For each irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} , $\text{mult}_{\mathcal{W}}(F) \leq 3m$, i.e., for general $p \in U$ we have $\text{mult}_W(\Delta_p) \leq 3$ for any irreducible component W of $\text{Supp}(\Delta_p)$ passing through p . After differentiating F , we will construct a

“well-behaved” covering family of tigers of dimension one. We will derive an upper bound of ω by studying this covering family of tigers. See Section 5.3.

- (2) (**Big multiplicity**) There exists an irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} with multiplicity $\text{mult}_{\mathcal{W}}(F) > 3m$, i.e., for general $p \in U$ we have $\text{mult}_W(\Delta_p) > 3$ for some irreducible component W of $\text{Supp}(\Delta_p)$ passing through p . We will construct a covering family of tigers of dimension two and derive an upper bound of ω by studying the geometry of this covering family of tigers. See Section 5.4.

To pick a “well-behaved” nonzero section in $H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$, we will apply the following proposition.

Proposition 5.2. ([20, Proposition 5.2.13]) *Let X and U be smooth irreducible varieties, with U affine, and suppose that $\mathcal{Z} \subseteq \mathcal{W} \subseteq X \times U$ are irreducible subvarieties such that \mathcal{W} dominates X . Fix a line bundle L on X , and suppose we are given a divisor $F \in |\text{pr}_X^*(L)|$ on $X \times U$. Write $l = \text{mult}_{\mathcal{Z}}(F)$ and $k = \text{mult}_{\mathcal{W}}(F)$. After differentiating in the parameter directions, there exists a divisor $F' \in |\text{pr}_X^*(L)|$ on $X \times U$ with the property that $\text{mult}_{\mathcal{Z}}(F') \geq l - k$, and $\mathcal{W} \not\subseteq \text{Supp}(F')$.*

5.3. Small multiplicity. Let X be an ϵ -klt \mathbb{Q} -Fano threefold of Picard number one and write $\text{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω . Denote $D = -2K_X$, we have $D^3 > (6\omega)^3$. By Section 5.1, there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \text{div}(s)$ on $X \times U$.

Proposition 5.3. *With the set up above. Assume that $\omega > 4$. If we are in the case where all the irreducible components \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} satisfy $\text{mult}_{\mathcal{W}}(F) \leq 3m$, then $\omega < 8/\epsilon + 4$. In particular, there is an upper bound for the volume*

$$\text{Vol}(-K_X) = -K_X^3 \leq \left(\frac{24}{\epsilon} + 12\right)^3.$$

Proof. Let M be the maximum of $\text{mult}_{\mathcal{W}}(F)$ among all the irreducible components \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} . Then $M \leq 3m$ by the hypothesis. For a fixed irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} , we can apply Proposition 5.2 to F . We obtain a divisor $F' \in |\text{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ with the property that

$$\text{mult}_{\mathcal{Z}}(F') \geq (6m - M) \geq 3m, \text{ and } \mathcal{W} \not\subseteq \text{Supp}(F').$$

Since there are only finitely many irreducible components of $\text{Supp}(F)$ passing through \mathcal{Z} , by taking a generic differentiation, it follows that for a general divisor $F'' \in |\text{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ we have $\mathcal{W} \not\subseteq \text{Supp}(F'')$ for any irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} . In particular, the base locus $\text{Bs}(|\text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of \mathcal{Z} .

Let G be a general divisor in $|\text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ and $\Delta_p = G_p/m$ for $p \in U$ general the corresponding \mathbb{Q} -divisors on X . It follows that $p \in \text{Nklt}(K_X + \Delta_p)$ as $\text{mult}_p \Delta_p \geq 3$. The minimal non-klt center $V_p \subseteq \text{Nklt}(K_X + \Delta_p)$ passing through p must be positive dimensional by Lemma 3.5 as the weight of Δ_p is $\omega/2 > 2$. Note that we may replace $|\text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ by $|\text{pr}_X^* L^{\otimes k} \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes k(6m-M)}|$ for any $k \geq 1$ and hence we may assume that $m \gg 0$. In particular, we have $0 \leq \text{mult}_W \Delta_p \ll 1$ for W any irreducible component of $\text{Supp}(\Delta_p)$, and V_p can only be one-dimensional.

Let $\pi : Y \rightarrow X$ and $f : Y \rightarrow B$ be a one dimensional covering family of tigers of weight $\omega' \geq \omega/2$ constructed from the Δ_p 's above by Lemma 3.2 and Lemma 3.3. By abuse of notation, we still denote Δ_p 's the divisors associated to this covering family of tigers.

Choose $p, q \in U \subseteq X$ general. By Lemma 1.3, the non-klt locus $\text{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q)) \supseteq V_p \cup V_q$ on Y is connected and it contains a one dimensional cycle $C_{p,q}$ connecting V_p and V_q . Since Y is normal, an irreducible component C of $C_{p,q}$ intersecting V_q satisfies $C \cap Y_{\text{sm}} \neq \emptyset$ for $p, q \in X$ general. Since C is in $\text{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q))$, by symmetry we have $\text{mult}_C(\pi^*(\Delta_p)) > \epsilon/2$.

Suppose that $\Sigma \subseteq \text{Supp}(\pi^*(\Delta_p))$ is an irreducible component containing C . If $f(\Sigma) = f(C)$ is a curve, then $V_p \subseteq \Sigma = f^{-1}(f(C))$ as the general fiber of $f : Y \rightarrow B$ is irreducible. Moreover, we can assume that Σ is not π -exceptional as there are only finitely many π -exceptional divisors and we choose $p \in X$, and hence V_p , general. Note that there can only be one such Σ once we fix $p \in X$ and C . In particular, $\Sigma \subseteq \text{Supp}(\pi_*^{-1}(\Delta_p))$ is an irreducible component containing V_p , and we can write $\pi^*(\Delta_p) = \Delta' + \lambda\Sigma$ with $\Delta' \wedge \Sigma = 0$. Moreover, $\lambda \leq 1/m$, where $m \gg 0$ by our choice of Δ_p with $0 \leq \text{mult}_W \Delta_p \ll 1$ for W any irreducible component of $\text{Supp}(\Delta_p)$. Also, $\text{mult}_C \Sigma = 1$ since Σ is smooth along C as $f(C)$ passes through a general point of B and Y is smooth in codimension one.

Choose a general point $b' \in f(C)$, we have that $Y_{b'}$ is a general fiber of $f : Y \rightarrow B$ and

$$\frac{2}{\frac{3}{2} - 2} \geq \frac{2}{\omega}(-K_X \cdot V_t) = \pi^*(\Delta_p) \cdot Y_{b'} = (\Delta' + \lambda\Sigma) \cdot Y_b > \frac{\epsilon}{2} - \frac{1}{m},$$

where the first inequality follows from Corollary 3.7. The second inequality follows from $\Sigma \cdot Y_b \geq 0$ and $\text{mult}_C \Delta' = \text{mult}_C(\pi^*(\Delta_p)) - \lambda \text{mult}_C \Sigma$. Since $m \gg 0$, we get $\omega \leq 8/\epsilon + 4$. \square

Remark 5.4. In the proof of Proposition 5.3, the difficulty arises because in general the one cycle C might be contained in $\text{Supp}(\pi_*^{-1}(\Delta_p))$. In this case, one can not see the contribution of the ϵ -klt condition from the intersection number $\pi^* \Delta_p \cdot Y_b$ for Y_b a general fiber over $f(C) \subseteq B$ as $Y_b \subseteq \text{Supp}(\pi_*^{-1}(\Delta_p))$, cf., Example 2.1. The differentiation argument eliminates the contribution of irreducible components of $\text{Supp}(\pi_*^{-1}(\Delta_p))$ along Y_b .

5.4. Big multiplicity. Again, let X be an ϵ -klt \mathbb{Q} -factorial \mathbb{Q} -Fano threefold of Picard number one. Write $\text{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω and denote $D = -2K_X$. As before, by Section 5.1, there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \text{div}(s)$ on $X \times U$. We now consider the case where there exists an irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} with multiplicity $\text{mult}_{\mathcal{W}}(F) > 3m$.

Lemma 5.5. *If there exists an irreducible component \mathcal{W} of $\text{Supp}(F)$ passing through \mathcal{Z} with multiplicity $\text{mult}_{\mathcal{W}}(F) > 3m$, then there exists a covering family of tigers of dimension two and weight $\omega' \geq \omega/2$.*

Proof. Fix \mathcal{W} to be one of these irreducible components of $\text{Supp}(F)$. We have the inclusions $\mathcal{Z} \subseteq \mathcal{W} \subseteq X \times U$ with the projection map $\mathcal{W} \rightarrow U$. Cutting down by hyperplanes on U and restricting to a smaller open subset of U , we may assume that $\mathcal{W} \rightarrow U$ factors through a Hilbert scheme of X and $\mathcal{W} \rightarrow X$ is generically finite. Replace U by the normalization of the closure of its image in the Hilbert scheme and \mathcal{W} by the normalization of universal family. We obtain maps $\pi : Y \rightarrow X$ and $f : Y \rightarrow B$. Note that a general fiber Y_b is two dimensional. We claim that the pairs $(\Delta_b = \pi_*(Y_b), V_b = Y_b)$ is a two dimensional covering of tigers of weight $\omega' \geq \omega/2$.

Since X is \mathbb{Q} -factorial and $\rho(X) = 1$, the integral divisor $\Delta_b = \pi_*(Y_b)$ for any $p \in B$ on X is \mathbb{Q} -linear equivalent to a multiple of $-K_X$. Since $\mathcal{W} \leq F$, we have $\pi_*(Y_b) \leq F_b$ for general $b \in B$. In particular, $\pi_*(Y_b) \sim_{\mathbb{Q}} -K_X/\omega'$ for some $\omega' \geq \omega/2$. Since any two general divisors $\pi_*(Y_{b_i})$, $i = 1, 2$, on X are \mathbb{Q} -linear equivalent as the base field is uncountable, and it is clear that $V_t = \pi(Y_b)$ is the minimal non-klt center of $\text{Nklt}(X, \Delta_b)$, and the lemma follows. \square

Let $\pi : Y \rightarrow X$ with $f : Y \rightarrow B$ be a covering family of tigers of dimension two and weight $\omega' \geq \omega/2$ given by Lemma 5.5. We first deal with case where $\pi : Y \rightarrow X$ is not birational.

Proposition 5.6. *Suppose that the two dimensional covering family of tigers $\pi : Y \rightarrow X$ with $f : Y \rightarrow B$ of weight $\omega' \geq \omega/2$ is not birational and assume that $\omega > 12$, then $\omega \leq 24/\epsilon + 12$. In*

particular, there is an upper bound of volume

$$\mathrm{Vol}(-K_X) = -K_X^3 \leq \left(\frac{72}{\epsilon} + 36\right)^3.$$

Proof. Let $d \geq 2$ be the degree of $\pi : Y \rightarrow X$. Fix an open subset $U \subseteq X$ such that for a general point $p \in U$ there are d divisors $\Delta_p^{t_i}$, for some $t_1, \dots, t_d \in B$, with $\pi(Y_{t_i}) \subseteq \mathrm{Nklt}(X, \Delta_p^{t_i})$ the unique minimal non-klt center passing through p . Consider the collection of \mathbb{Q} -divisors $\{\Delta'_p = \frac{6}{d} \sum_{i=1}^d \Delta_p^{t_i} | p \in U\}$, then $\mathrm{mult}_p \Delta'_p \geq 6$, $\mathrm{mult}_{W'} \Delta'_p = \frac{6}{d} \leq 3$ for $W' \subseteq \mathrm{Supp}(\Delta'_p)$ any irreducible component, and $\Delta'_p \sim_{\mathbb{Q}} \frac{-K_X}{d\omega'/6}$.

By the same construction as in Section 5.1, possibly after shrinking U to a smaller open affine subset, there exists an integer $m > 0$ such that $H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ where $L = 6m(-K_X)/d\omega'$ is Cartier. Let $t \in H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a general nonzero section and $G = \mathrm{div}(t)$ be the associated divisor on $X \times U$. Note that $\mathrm{mult}_{\mathcal{Z}}(G) \geq 6m$ and $\mathrm{mult}_{\mathcal{W}}(G) \leq 6m/d \leq 3m$ for any irreducible component \mathcal{W} of $\mathrm{Supp}(G)$ passing through \mathcal{Z} . Indeed, we know that for general $p \in U$ there is the divisor Δ'_p with $\mathrm{mult}_p \Delta'_p \geq 6$ and $\mathrm{mult}_{W'} \Delta'_p = \frac{6}{d} \leq 3$ for any irreducible component $W' \subseteq \mathrm{Supp}(\Delta'_p)$. Since t is a general section, $t_p = t|_{X \times \{p\}}$ is also a general section for general $p \in U$. Using Lemma 5.1 to compute the multiplicity, we obtain $\mathrm{mult}_{\mathcal{W}}(G) = \mathrm{mult}_{\mathcal{W}_p}(G_p) \leq m \cdot \mathrm{mult}_{W'} \Delta'_p \leq 3m$, where $G_p = \mathrm{div}(t_p)$ and \mathcal{W}_p is any irreducible component of $\mathrm{Supp}(G_p)$.

By a differentiation argument and the same construction as in Proposition 5.3, there is a covering family of tigers (Δ_t, V_t) of dimension one and weight $\omega'' \geq d\omega'/6 \geq d\omega/12$, which satisfies the property that the base locus $\mathrm{Bs}(|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of \mathcal{Z} , where M is the maximum of $\mathrm{mult}_{\mathcal{W}}(G)$ amongst all the irreducible components \mathcal{W} of $\mathrm{Supp}(G)$ passing through \mathcal{Z} . Hence by Corollary 3.7, we get

$$\frac{2}{\omega'' - 2} \geq \frac{1}{\omega''} (-K_X \cdot V_t) = \pi^* \Delta_p \cdot Y_b \geq \frac{\epsilon}{2}.$$

In particular,

$$\frac{4}{\epsilon} + 2 \geq \omega'' \geq \frac{d\omega}{12} \geq \frac{\omega}{6},$$

and $\omega \leq 24/\epsilon + 12$. □

Assumption. From now on, we assume that $\pi : Y \rightarrow X$ with $f : Y \rightarrow B$ is a **birational** covering family of tigers of dimension two and weight $\omega' \geq \omega/2$. Write $K_Y + \Gamma - R = \pi^* K_X$ where Γ and R are effective divisors on Y with no common components.

Lemma 5.7. *There is a π -exceptional divisor E on Y dominating B . In particular, $\pi : Y \rightarrow X$ is not small.*

Proof. Suppose that there is no π -exceptional divisors dominating B . Let A_B be a sufficiently ample divisor on B and $A_Y = f^* A_B$ the pull-back. Since $\rho(X) = 1$, the divisor $A_X = \pi_* A_Y$ on X is ample and $\pi^* A_X = A_Y + G$ for some effective π -exceptional divisor G . By assumption $f(G) \subseteq B$ has codimension one and hence $A_Y + G \leq f^* H$ for some divisor H on B . This is a contradiction since then $A_Y + G$ is not big but $\pi^* A_X$ is. □

The following lemma is crucial for computing the restricted volume. The key point is that it allows us to control the negative part of the subadjunction $-K_X|_{V_t}$. Note that the proof fails in higher dimensions, cf. [22, Lemma 6.2].

Lemma 5.8. *Let E be a π -exceptional divisor dominating B . For general points $p, q \in X$ we have that $E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_p + \Delta_q))$. In particular, denote $H = \pi^*(-K_X)$. For any π -exceptional divisor E dominating B we have*

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \epsilon E.$$

Proof. Since the construction of covering families of tigers is done via the Hilbert scheme, π is finite on the general fibers V_t of $f : Y \rightarrow B$. Recall that $\pi(V_t) \subseteq X$ is the minimal non-klt center of $(X, \Delta_{p(t)})$ for some $\Delta_{p(t)}$ passing through a general point $p(t) \in X$. We denote $\Delta_{p(t)}$ by Δ_t for simplicity.

Let E be a π -exceptional divisor dominating B . Since $E \cap V_b$ is one dimensional for general $b \in B$ and $\pi|_{V_b}$ is finite, $\dim \pi(E) > 0$ as $\pi(E) \supseteq \pi(E \cap V_b)$. Since E is irreducible and π -exceptional, $\pi(E)$ is an irreducible curve. Fix $t_1, t_2 \in B$ two general points. Pick a general point $x \in \pi(E)$ and consider its preimage on V_{t_i} . Since π is finite on the general fiber V_t , $\pi^{-1}(x) \cap V_{t_i}$ can only be a discrete finite set. Choose $x_i \in \pi^{-1}(x) \cap V_{t_i}$ over x for $i = 1, 2$. Apply the Connectedness Lemma 1.3 to the pair $(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ over X . There is a (possibly reducible) curve contained in $\pi^{-1}(x) \cap \text{Nklt}(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ connecting x_1 and x_2 . The component of this curve containing x_1 can not lie on V_{t_1} as the map π is finite on V_{t_1} . As $x \in \pi(E)$ is general, this curve deforms into a dimension two subset of E by moving $x \in \pi(E)$. Since E is irreducible, the closure of this two dimensional subset coincides with E and hence $E \subseteq \text{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$. In particular, $\text{mult}_E(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2})) \geq 1$. If $E \not\subseteq \text{Supp}(\Gamma)$, then $\pi^*(\Delta_p + \Delta_q) \geq E$. If $E \subseteq \text{Supp}(\Gamma)$, then $\pi^*(\Delta_p + \Delta_q) \geq \epsilon E$ since $\Gamma \in [0, 1 - \epsilon]$ as X is ϵ -klt. \square

To study the geometry of the covering family $f : Y \rightarrow B$, we would like to run a relative minimal model program of (Y, Γ) over B . However, Y is normal but possibly not \mathbb{Q} -factorial. To get a \mathbb{Q} -factorial model of (Y, Γ) , we adopt Hacon's dlt models, cf. [15, Theorem 3.1]. In fact, since the volume bound will be obtained by doing a computation on a general fiber Y_b , it suffices to modify Y over an open subset $U \subseteq B$.

Lemma 5.9. *After restricting to an open subset $U \subseteq B$ and replacing Y by a suitable birational model, we can assume that Y is \mathbb{Q} -factorial and (Y, Γ) is $\epsilon/2$ -klt. Moreover, we can assume for E any π -exceptional divisor dominating U and $p, q \in X$ general, we have that*

$$(5.1) \quad \frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} E.$$

Proof. Fix $p, q \in X$ general and consider the pair

$$(\sharp) \quad K_Y + \Gamma - R_d + \pi^*(\Delta_p + \Delta_q) - R_e \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_p + \Delta_q)$$

where $R = R_d + R_e$ with $(-)_d$ the sum of components dominating B and $(-)_e$ the sum of components mapping to points in B . Restricting Y to $Y_U = f^{-1}(U)$ for a suitable nonempty open set $U \subseteq B$, we may assume that $R_e = 0$ and (\sharp) becomes

$$K_Y + \Gamma - R_d + \pi^*(\Delta_p + \Delta_q) \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_p + \Delta_q).$$

We abuse the notation: Y is understood to be Y_U if not specified.

Denote $\Gamma_{p,q} = \Gamma - R_d + \pi^*(\Delta_p + \Delta_q)$. Note that $\Gamma_{p,q} \geq 0$ by Lemma 5.8. Let $\phi : W \rightarrow Y$ be a log resolution of $(Y, \Gamma_{p,q})$ and write

$$K_W + \phi_*^{-1} \Gamma_{p,q} + Q \sim_{\mathbb{Q}} \phi^*(K_Y + \Gamma_{p,q}) + P,$$

where $Q, P \geq 0$ are ϕ -exceptional divisors with $Q \wedge P = 0$. We aim to modify W by running a relative minimal model program over Y with scaling of an ample divisor so that it contracts $Q^{<1-\epsilon/2} + P$, where $(\sum_i a_i Q_i)^{<\alpha} := \sum_{a_i < \alpha} a_i Q_i$. Note that we define $(-)^{\alpha \leq \cdot < \beta}$ and $(-)^{\geq \alpha}$ in the same way.

Consider $F = \sum_i F_i$, where the sum runs over all the ϕ -exceptional divisors with log discrepancy in $(\epsilon/2, 1]$ with respect to $(Y, \Gamma_{p,q})$, then

$$(F + P) \wedge Q^{\geq 1-\epsilon/2} = 0, \text{ and } \text{Supp}(F) \supseteq \text{Supp}(Q^{<1-\epsilon/2}).$$

Since $(Y, \Gamma - R)$ is ϵ -klt, the divisor Γ on Y as well as $\phi_*^{-1}\Gamma$ on W has coefficients in $[0, 1 - \epsilon]$. For rational numbers $0 < \epsilon < \epsilon' < 1$ and $0 < \delta, \delta' \ll 1$, we have the following $\epsilon/2$ -klt pair

$$\begin{aligned} & K_W + \phi_*^{-1}\Gamma + Q^{<1-\epsilon/2} + \delta'Q^{1-\epsilon/2 \leq \cdot < 1} + (1 - \epsilon')(Q^{\geq 1})_{\text{red}} + \delta F \\ & \sim_{\mathbb{Q}} \phi^*(K_Y + \Gamma_{p,q}) - (\phi_*^{-1}\Gamma_{p,q} - \phi_*^{-1}\Gamma) - (1 - \delta')Q^{1-\epsilon/2 \leq \cdot < 1} - (Q^{\geq 1} - (1 - \epsilon')(Q^{\geq 1})_{\text{red}}) \\ & \quad + P + \delta F \end{aligned}$$

where $(\sum_j b_j G_j)_{\text{red}} := \sum_{b_j \neq 0} G_j$. We denote the above pair by (W, Ξ) where

$$\Xi = \phi_*^{-1}\Gamma + Q^{<1-\epsilon/2} + \delta'Q^{1-\epsilon/2 \leq \cdot < 1} + (1 - \epsilon')(Q^{\geq 1})_{\text{red}} + \delta F.$$

By [7], a relative minimal model program with scaling of an ample divisor of the pair (W, Ξ) over Y terminates with a birational model $\psi : W \dashrightarrow W'$ over Y with $\phi' : W' \rightarrow Y$ the induced map. We obtain the following diagram,

$$\begin{array}{ccccc} & & & W' & \xleftarrow{\psi} W \\ & \nearrow \pi' & & \downarrow \phi' & \nearrow \phi \\ X & \xleftarrow{\pi} Y & \longleftrightarrow & Y_U & \\ & \downarrow f & & \downarrow & \\ & B & \longleftrightarrow & U & \end{array}$$

where $\pi' : W' \rightarrow X$ is the induced map.

Write $K_{W'} + \Gamma_{W'} - R_{W'} \sim_{\mathbb{Q}} \pi'^* K_X$ where $\pi' = \phi' \circ \pi$. Note that $\Gamma_{W'} \in [0, 1 - \epsilon]$ by the ϵ -klt condition and $\Gamma_{W'} - (\phi')_*^{-1}\Gamma \geq 0$ is ϕ' -exceptional. It follows by the construction that $\Gamma_{W'} \leq \psi_* \Xi$. In particular, $(W', \Gamma_{W'})$ is $\epsilon/2$ -klt as the pair (W, Ξ) is $\epsilon/2$ -klt and the minimal model program does not make singularities worse.

On W' , the divisor

$$G = \psi_*(-(\phi_*^{-1}\Gamma_{p,q} - \phi_*^{-1}\Gamma) - (1 - \delta')Q^{1-\epsilon/2 \leq \cdot < 1} - (Q^{\geq 1} - (1 - \epsilon')(Q^{\geq 1})_{\text{red}}) + P + \delta F)$$

is ϕ' -nef with $\phi'_* G \leq 0$ since $\Gamma_{p,q} \geq \Gamma$. By [19, Negativity Lemma 3.39], we have that $G \leq 0$. Since $F + P$ is ϕ -exceptional and $(F + P) \wedge Q^{\geq 1-\epsilon/2} = 0$, it follows that $\psi_*(P + \delta F) = 0$. In particular, all the ϕ' -exceptional divisors on W' have log discrepancies less than or equal to $\epsilon/2$ with respect to $(Y, \Gamma_{p,q})$.

We now show that for any π' -exceptional divisor E' on W' dominating U , E' satisfies the inequality

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} E',$$

where $H' = \pi'^*(-K_X)$. This is easy to see. If $E = \phi'_*(E') \neq 0$ on Y_U , then by Lemma 5.8, $E \subseteq \text{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_p + \Delta_q))$ and hence $E' \subseteq \text{Nklt}(K_{W'} + \Gamma_{W'} - R_{W'} + \pi'^*(\Delta_p + \Delta_q))$. The inequality then follows from the same argument as in Lemma 5.8. If $\phi'_* E' = 0$, then by construction $\text{mult}_{E'}(K_{W'} + \Gamma_{W'} - R_{W'} + \pi'^*(\Delta_p + \Delta_q)) \geq 1 - \epsilon/2$. Suppose that $E' \subseteq \text{Supp}(R_{W'})$, then

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq E' \geq \frac{\epsilon}{2} E'.$$

If $E' \subseteq \text{Supp}(\Gamma_{W'})$, then as $\Gamma_{W'} \in [0, 1 - \epsilon]$ we get

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq ((1 - \frac{\epsilon}{2}) - (1 - \epsilon))E' = \frac{\epsilon}{2} E'.$$

It follows that W' satisfies the required properties. \square

Remark 5.10. Write $\Gamma = \pi_*^{-1}\Delta + \Gamma_d + \Gamma_e$ and $R = R_d + R_e$, where $(-)_d$ is the sum of components dominating B and $(-)_e$ is the sum of components mapping to points in B . From the proof of Lemma 5.9, we deduce the following two inequalities :

$$(5.2) \quad \frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq R_d \text{ and } \frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} \Gamma_d.$$

Now let $\pi : Y \rightarrow X$ with $f : Y \rightarrow U$ be the modified birational covering family of tigers of dimension two and weight $\omega' \geq \omega/2$ given by Lemma 5.9, where Y is now \mathbb{Q} -factorial. Write $K_Y + \Gamma - R \sim_{\mathbb{Q}} \pi^* K_X$, where $\Gamma, R \geq 0$ are π -exceptional and $\Gamma \wedge R = 0$. The pair (Y, Γ) is $\epsilon/2$ -klt with $\Gamma \in [0, 1 - \epsilon/2)$ and note that $H = \pi^*(-K_X)$ is semi-ample and big on Y .

Recall that for a projective morphism $\phi : Z \rightarrow U$, a divisor D on Z is pseudo-effective (PSEF) over U if the restriction of D to the generic fiber is pseudo-effective.

Lemma 5.11. *Assume that $\omega' > 2$ and consider the pseudo-effective threshold of $K_Y + \Gamma$ over U with respect to H*

$$\tau := \inf\{t > 0 \mid K_Y + \Gamma + tH \text{ is PSEF over } B\}.$$

Then $1 \geq \tau \geq 1 - \frac{2}{\omega'} > 0$.

Proof. Since $K_Y + \Gamma + H \sim_{\mathbb{Q}} R \geq 0$, the first inequality is clear. Restricting to a general fiber Y_u of Y over U , we have

$$\begin{aligned} (K_Y + \Gamma + \tau H)|_{Y_u} &= (R - (1 - \tau)H)|_{Y_u} \\ &= (R_d - \frac{2}{\omega'} H)|_{Y_u} - (1 - \tau - \frac{2}{\omega'}) H|_{Y_u} \end{aligned}$$

which can not be PSEF if $\omega' > 2$ and $\tau < 1 - \frac{2}{\omega'}$ since the first term is non-positive by (5.2) and the second term is negative. \square

Now we run a relative minimal model program with scaling for the covering family of tigers $f : Y \rightarrow U$. Since (Y, Γ) is $\epsilon/2$ -klt and H is semiample and big, we may assume that $(Y, \Gamma + \tau' H)$ remains $\epsilon/2$ -klt for any rational number $0 < \tau' < \tau$. By [7], a relative minimal model program of $(K_Y + \Gamma + \tau' H)$ with scaling of H over U terminates with a relative Mori fiber space $Y' \rightarrow T$ over U with $\dim Y' > \dim T \geq \dim U$. Denote the induced maps by $g : Y \dashrightarrow Y'$, $\psi : Y' \rightarrow T$, and $\phi : Y' \rightarrow U$. We obtain the following diagram,

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & Y & \dashrightarrow^g & Y' \\ & & \downarrow f & \swarrow \phi & \downarrow \psi \\ & & B \supseteq U & \xleftarrow{\quad} & T. \end{array}$$

For a general fiber Y'_t of $\psi : Y' \rightarrow T$, by construction, the Picard number $\rho(Y'_t) = 1$ and the divisor $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ on Y'_t is ample.

Lemma 5.12. *There exists a divisor E' on Y' which is exceptional over X and dominates T .*

Proof. Recall that there is a natural map $T \rightarrow U \rightarrow B$. We can extend $\psi : Y' \rightarrow T$ to $\bar{\psi} : \bar{Y}' \rightarrow \bar{T}$ over B where $\bar{(-)}$ stands for a projective compactification of $(-)$. Take a common resolution $p : W \rightarrow X$ and $q : W \rightarrow \bar{Y}'$ and let $A_{\bar{T}}$ be a sufficiently ample divisor on \bar{T} . Let $A_{\bar{Y}'} = \bar{\psi}^* A_{\bar{T}}$, $A_W = q^* A_{\bar{Y}'}$, and $A_X = p_* A_W$. Then $p^* A_X = A_W + E = q^* A_{\bar{Y}'} + E = q^* \bar{\psi}^* A_{\bar{T}} + E$ for an effective divisor E on W which is exceptional over X . Since $\rho(X) = 1$, it follows by the same argument as in Lemma 3.2 that one of the irreducible components of E maps to a divisor E' on \bar{Y}' . By the same argument as in Lemma 3.2 again, one of the irreducible components of the nonzero divisor $q_*(E)$ dominates \bar{T} . \square

Proposition 5.13. *If $\dim T = 2$, then $\omega' \leq 8/\epsilon + 2$.*

Proof. By Lemma 5.12, there exists a divisor E' on Y' which is exceptional over X and dominates T . Note that Y' is normal and hence $\psi(\text{Sing}(Y'))$ is a proper subset of T . In particular, a general fiber Y'_t of $\psi : Y' \rightarrow T$ is a smooth projective curve and hence $E'.Y'_t \geq 1$. Since the divisor $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ is ample, a general fiber Y'_t is a smooth rational curve \mathbb{P}^1 . From (5.1), we know that

$$\frac{2}{\omega'}H' - \frac{\epsilon}{2}E' \sim_{\mathbb{Q}} \text{effective}.$$

Also from (5.2),

$$\begin{aligned} -(K_{Y'} + \Gamma').Y'_t &= (H' - R').Y'_t = (1 - \frac{2}{\omega'})H'.Y'_t + (\frac{2}{\omega'}H - R').Y'_t \\ &\geq (1 - \frac{2}{\omega'})H'.Y'_t. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{2}{\omega'} &\geq \frac{1}{\omega'}(-(K_{Y'} + \Gamma').Y'_t) \geq \frac{1}{\omega'}(1 - \frac{2}{\omega'})H'.Y'_t \\ &\geq (1 - \frac{2}{\omega'})\frac{\epsilon}{4}E'.Y'_t \\ &\geq (1 - \frac{2}{\omega'})\frac{\epsilon}{4} \end{aligned}$$

where the first inequality follows by the adjunction formula on \mathbb{P}^1 . Hence $\omega' \leq \frac{8}{\epsilon} + 2$. \square

Proposition 5.14. *If $\dim T = 1$, then*

$$\omega' \leq \frac{4M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 2$$

where $R(2, \epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$ and $M(2, \epsilon)$ is an upper bound of the volume $\text{Vol}(-K_S) = K_S^2$ for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$.

Proof. Since $f : Y \rightarrow U$ has connected fibers, $T \cong U$. Since $-(K_{Y'} + \Gamma'_d)|_{Y'_u} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_u}$ is ample and $\rho(Y'_u) = 1$ for a general point $u \in U$, we see that

$$-K_{Y'_u} \sim_{\mathbb{Q}} (H' + \Gamma'_d - R_d)|_{Y'_u}$$

is ample. By Lemma 5.12, let E' be a divisor on Y' exceptional over X which dominates U , then

$$-K_{Y'_u} \equiv (H' + \Gamma'_d - R_d)|_{Y'_u} \geq (1 - \frac{2}{\omega'})H|_{Y'_u} \geq (1 - \frac{2}{\omega'}) \cdot \frac{\omega'\epsilon}{4}E'_u$$

where the second inequality follows by dropping Γ'_d and applying (5.2) while the last one from (5.1). By intersecting with the ample divisor $-K_{Y'_u}$, this implies that

$$(-K_{Y'_u})^2 \geq (\omega' - 2)\frac{\epsilon}{4}E'_u.(-K_{Y'_u}).$$

Now (Y'_u, Γ'_u) is an $\epsilon/2$ -klt log del-Pezzo surfaces of Picard number one. Hence Y'_u is an $\epsilon/2$ -klt del-Pezzo surface of Picard number $\rho(Y'_u) = 1$. By Theorem 4.3, $(-K_{Y'_u})^2$ is bounded above by a positive number $M(2, \epsilon)$ satisfying

$$M(2, \epsilon) \leq \max\{64, \frac{16}{\epsilon} + 4\}.$$

Also, by (\diamond) the Cartier index of $K_{Y'_u}$ has an upper bounded

$$R(2, \epsilon) \leq r(2, \frac{\epsilon}{2}) \leq 2(4/\epsilon)^{128 \cdot 2^5 / \epsilon^5}.$$

It follows that

$$M(2, \epsilon) \geq (-K_{Y'_u})^2 \geq \frac{1}{R(2, \epsilon)}(\omega' - 2) \frac{\epsilon}{4} E'_u \cdot (\text{Ample Cartier}) \geq \frac{1}{R(2, \epsilon)}(\omega' - 2) \frac{\epsilon}{4}$$

and hence we get an upper bound

$$\omega' \leq \frac{4M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 2.$$

□

Remark 5.15. It has been shown in [6] that a klt log del Pezzo surface has at most four isolated singularities. Also surface klt singularities are classified by Alexeev in [1]. Hence we expect that it is possible to obtain a better upper bound for $R(2, \epsilon)$ and $M(2, \epsilon)$ in Proposition 5.14.

Theorem 5.16. *Let (X, Δ) be an ϵ -klt log \mathbb{Q} -Fano threefold of $\rho(X) = 1$. Then the degree $-K_X^3$ satisfies*

$$-K_X^3 \leq \left(\frac{24M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 12 \right)^3$$

where $R(2, \epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$ and $M(2, \epsilon)$ is an upper bound of the volume $\text{Vol}(S) = K_S^2$ for S any $\epsilon/2$ -klt log del Pezzo surface of $\rho(S) = 1$. Note that we have $M(2, \epsilon) \leq \max\{64, 16/\epsilon + 4\}$ from Theorem 4.3 and $R(2, \epsilon) \leq 2(4/\epsilon)^{128 \cdot 2^5 / \epsilon^5}$ from (\diamond) .

Proof. Recall that $\omega' \geq \omega/2$. The theorem then follows from Propositions 5.3, 5.13 and 5.14. □

The following example shows that the cone construction analogous to Example 2.1 only provides ϵ -klt Fano threefolds with volumes of order $1/\epsilon^2$.

Example 5.17. (Projective cone of projective spaces) For $n \geq 1$ and $d \geq 2$, let $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the embedding by $|\mathcal{O}(d)|$ and X be the associated projective cone. The projective variety X is normal \mathbb{Q} -factorial of Picard number one with unique singularity at the vertex O . Also, X admits a resolution $\pi : Y = \text{Bl}_O X \rightarrow X$ with exceptional divisor $E \cong \mathbb{P}^n$ of normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$. The variety Y is the projective bundle $\mu : Y \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)) \rightarrow \mathbb{P}^n$ with tautological bundle $\mathcal{O}_Y(1) \cong \mathcal{O}_Y(E)$. We have:

- $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ and hence $E^{n+1} = (-d)^n$;
- $K_Y = \pi^* K_X + (-1 + \frac{n+1}{d})E$ and hence X is always klt. Also, X is terminal (resp. canonical) if and only if $n+1 > d \geq 2$ (resp. $n+1 \geq d \geq 2$);
- $K_Y = \mu^*(K_{\mathbb{P}^n} + \det(\mathcal{E})) \otimes \mathcal{O}_Y(-\text{rk}(\mathcal{E})) \equiv -(n+1+d)F - 2E$ where the vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)$ and $F = \mu^* \mathcal{O}_{\mathbb{P}^n}(1)$;
- $F^{n+1} = 0$ and $F^{n+1-k} \cdot E^k = (-d)^{k-1}$ for $1 \leq k \leq n+1$;
- $K_Y^{n+1} = K_X^{n+1} + (-1 + \frac{n+1}{d})^{n+1} E^{n+1}$ and

$$\begin{aligned} K_Y^{n+1} &= \frac{-1}{d} \sum_{k=1}^{n+1} \binom{n+1-k}{k} \left(-1 + \frac{n+1}{d} \right)^{n+1-k} (2d)^k \\ &= \frac{-1}{d} ((d-n-1)^{n+1} - (-(d+n+1)^{n+1})); \end{aligned}$$

- In summary, $-K_X$ is ample with

$$(-K_X)^{n+1} = \frac{(d+n+1)^{n+1}}{d}.$$

If $n = 2$, then we have an ϵ -klt Fano threefold of Picard number one with $\epsilon = 1/d$. The volume $\text{Vol}(X) = (-K_X)^3$ is of order $1/\epsilon^2$.

In view of Theorem 5.16, it is then interesting to see whether ϵ -klt Fano threefolds with big volumes exist.

Question 5.18. *Can one find ϵ -klt \mathbb{Q} -factorial \mathbb{Q} -Fano threefolds X of $\rho(X) = 1$ with volume $\text{Vol}(X) = (-K_X)^3 = O(\frac{1}{\epsilon^c})$ for $c \geq 3$?*

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